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On a one-dimensional version of the dynamical Marguerre-Vlasov system

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— Dedicated to Constantine Dafermos on his 60^{th} birthday

Abstract. A one-dimensional version of the so-called Marguerre-Vlasov system of equations describing the vibrations of shallow shells is considered. The system depends on a parameter $\epsilon \to 0$ in a singular way and undergoes the effect of damping mechanisms. We show that the system converges to a nonlinear beam equation while the energy decays exponentially uniformly (on $\epsilon \to 0$) as time goes to infinity.

Keywords: singular limit, Marguerre-Vlasov system, uniform stabilization.

Mathematical subject classification: 35B40, 73C50, 73K15.

1 Introduction

We consider a one-dimensional version of the so-called dynamical Marguerre-Vlasov system which describes the vibrations of shallow shells (see [8] and [9]).

The damped one-dimensional system reads as follows

$$\begin{cases} u_{tt} = \dfrac{2}{1-\mu}\left[u_x + \dfrac{1}{2}w_x^2 + k_1(x)w\right]_x - u_t \\ w_{tt} + w_{xxxx} - w_{xxtt} = [f(u,w)]_x - g(u,w) - w_t + w_{xxt} \end{cases} \tag{1.1}$$

where

$$f(u,w) = \frac{2}{1-\mu}\left[w_x\left(u_x + \frac{1}{2}w_x^2 + k_1(x)w\right)\right] \tag{1.2}$$

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and

$$g(u, w) = \frac{2k_1}{1-\mu}\left[u_x + \frac{1}{2}w_x^2 + k_1(x)w\right]. \tag{1.3}$$

In (1.1), the space variable x runs in the interval $0 < x < L$ and t denotes the (positive) time variable. The quantities $u = u(x, t)$ and $w = w(x, t)$ represent, respectively, the longitudinal and transversal displacements of the beam at the point x at time t. Additionally, μ is a constant, $0 < \mu < 1$ and $k_1 = k_1(x)$ represents the curvature of the beam at the point x.

The terms $-u_t$ (resp. $-w_t + w_{xxt}$) of the first (resp. second equation) in (1.1) constitute damping mechanisms that dissipate the energy of solutions as time increases.

This work is devoted to analyze the following two questions:

a) Under a suitable perturbation of the system above in which the various constants are conveniently scaled, we investigate the proximity of the component w in (1.1) to the solution $z = z(x, t)$ of a scalar beam equation of Timoshenko's type.

b) The uniform (with respect to $\epsilon \to 0$) rate of decay of the total energy of the solutions of (1.1) as $t \to +\infty$.

To be more precise, given $\epsilon > 0$ and $0 \leq \alpha \leq 1$ we consider $u = u^\epsilon$, $w = w^\epsilon$ the solution of the coupled system of equations

$$\begin{cases} \epsilon u_{tt} = \dfrac{2}{1-\mu}\left[u_x + \dfrac{1}{2}w_x^2 + k_1(x)w\right]_x - \epsilon^\alpha u_t \\ w_{tt} + w_{xxxx} - w_{xxtt} = [f(u, w)]_x - g(u, w) - w_t + w_{xxt} \end{cases} \tag{1.4}$$

where f and g are given as in (1.2) and (1.3).

Once again, in (1.4) the variable x runs in the interval $0 < x < L$ and $t > 0$. We consider (1.4) with Dirichlet boundary conditions on u and clamped ends for w:

$$u(0, t) = u(L, t) = 0, \quad \forall t > 0 \tag{1.5}$$
$$w(0, t) = w(L, t) = w_x(0, t) = w_x(L, t) = 0, \quad \forall t > 0$$

and initial conditions at $t = 0$:

$$(u(0), u_t(0), w(0), w_t(0)) = (u_0, v_0, w_0, w_1) \in H, \tag{1.6}$$

where H is the *energy space*

$$H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I),$$

with $I = (0, L)$.

Problem (1.4)-(1.6) is globally well posed in the above space provided $k_1 \in H^1(I)$.

Moreover, the total energy associated with (1.4)-(1.6) is given by

$$E_\epsilon(t) = \frac{1}{2}\int_0^L \left[\epsilon u_t^2 + \frac{2}{1-\mu}\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 + w_t^2 + w_{xt}^2 + w_{xx}^2\right]dx, \quad (1.7)$$

and it is dissipated according to the law

$$\frac{d}{dt}E_\epsilon(t) = -\int_0^L \left[\epsilon^\alpha u_t^2 + w_t^2 + w_{xt}^2\right]dx. \quad (1.8)$$

According to this, in particular, $w = w^\epsilon$ in uniformly bounded in $L^\infty(0, \infty; H_0^2(0, L))$.

The first result of this paper guarantees that, as $\epsilon \to 0$, the component w^ϵ of the solution converges in the weak-* topology of that space to the solution z of the equation

$$z_{tt} + z_{xxxx} - z_{xxtt} = h(t)z_{xx} - z_t + z_{xxt} - k_1 h(t) \quad (1.9)$$

where

$$h(t) = \frac{1}{1-\mu}\left[\frac{1}{L}\int_0^L \left(z_x^2 + 2k_1 z\right)dx\right] \quad (1.10)$$

together with the boundary and initial conditions

$$\begin{cases} z(0,t) = z(L,t) = z_x(0,t) = z_x(L,t) = 0 \quad \forall t > 0 \\ z(x,0) = w_0(x), z_t(x,0) = w_1(x), \quad 0 < x < L. \end{cases} \quad (1.11)$$

In what concerns the second question related to the uniform decay rate of solutions, we prove that there exist positive constants $c > 0$ and $\beta > 0$ such that.

$$E_\epsilon(t) \le C E_\epsilon(0) \exp\left(-\frac{\beta t}{1 + \epsilon^\alpha\left[E_\epsilon(0) + \|k_1\|_\infty^2\right]}\right) \quad (1.12)$$

for all $t \ge 0$ where $0 \le \alpha \le 1$.

These problems have been previously considered by the authors in [6] (together with A. Pazoto) and [7] in the context of the classical von Kármán system for the vibrations of a beam. There, it was proved that:

a) Timoshenko's beam model may be derived as a singular limit of the Von Kármán beam model,

b) A similar uniform (as $\epsilon \to 0$) exponential decay rate as $t \to \infty$ of the energy of solutions holds.

Therefore, in this paper we extend these results to the 1-D model of the so-called Marguerre-Vlasov system for shallow shells.

As far as we know, model (1.9), which is a ''perturbed'' Timoshenko's type equation, has not been studied before. However, it can be easily handeled by the by now classical methods, as a perturbation of the classical Timoshenko beam equation.

Our notations are standard and can be found in the book of J.L.Lions [4]

2 Global well-posedness

In this section, for the sake of completeness we analyse the problem of the existence and uniqueness of solutions of system (1.4)-(1.6).

Let $\epsilon > 0$, $0 < \mu < 1$ and $\alpha \geq 0$ and consider the Hilbert space

$$H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I)$$

where $I = \{0 < x < L\}$ endowed with the norm

$$\|(u, y, w, p)\|_H^2 = \frac{2}{1-\mu}\|u_x\|^2 + \varepsilon\|y\|^2 + \|w_{xx}\|^2 + \|p\|^2 + \|p_x\|^2$$

for any $(u, y, w, p) \in H$. Here $\| \cdot \|$ denotes the norm in $L^2(I)$.

We write problem (1.4)-(1.6) in the abstract form

$$\begin{cases} DU_t = AU + N(U) \\ U(0) = U_0 = (u_0, u_1, w_0, w_1) \in H \end{cases} \tag{2.1}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{1-\mu}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix}$$

$$N(U) = \begin{bmatrix} 0 \\ \frac{2}{1-\mu}\left(\frac{1}{2}w_x^2 + k_1 w\right)_x - \epsilon^\alpha y \\ 0 \\ m(U) \end{bmatrix}$$

where

$$m(U) = \frac{2}{1-\mu}\left[w_x\left(4_x + \frac{1}{2}w_x^2 + k_1 w\right)\right]_x - \frac{2k_1}{1-\mu}\left[u_x + \frac{1}{2}w_x^2 + k_1 w\right] - w_t + w_{xxt}$$

and $U = (u, y, w, p)^\tau$.

The operator $\widetilde{A} = D^{-1}A$ with domain $D(\widetilde{A}) = (H^2 \cap H_0^1)(I) \times H_0^1(I) \times (H^3 \cap H_0^2)(I) \times H_0^2(I)$ is the infinitesimal generator of a semigroup of operators in H.

A direct calculation shows that, for any $U \in D(\widetilde{A})$ we have that

$$\langle \widetilde{A}U, U \rangle_H = \frac{2}{1-\mu}(y_x, u_x) + \frac{2}{1-\mu}(u_{xx}, y) + (p_{xx}, w_{xx})$$

$$-\left(\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w, p\right) - \left(\frac{\partial}{\partial x}\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w, p_x\right)$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(I)$.

Integrating by parts and observing that the term (p_{xx}, w_{xx}) can be written as $\left(p, \left(I - \frac{\partial^2}{\partial x^2}\right)\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial^4}{\partial x^4} w\right)$ we get

$$\langle \widetilde{A}U, U \rangle_H = 0. \tag{2.2}$$

Now, given $G = (g_1, g_2, g_3, g_4)^\tau \in H$, we claim that the system

$$\widetilde{A}U = G \tag{2.3}$$

admits a unique solution $U \in D(\widetilde{A})$. This is equivalent to finding $(u, y, w, p) \in H$ such that

a) $y = g_1$,

b) $\frac{2}{(1-\mu)\epsilon} u_{xx} = g_2$ with $u(0) = u(L) = 0$,

c) $p = g_3$

and

d) $-\left(I - \frac{\partial^2}{\partial_x 2}\right)^{-1} \frac{\partial^4}{\partial_x^4} w = g_4$ with $w = w_x = 0$ at $x = 0, L$.

Clearly, b) admits a unique solution $u \in H^2 \cap H_0^1(I)$ since $g_2 \in L^2(I)$. Problem d) is equivalent to

$$\frac{\partial^4}{\partial x^4} w = -\left(I - \frac{\partial^2}{\partial x^2}\right) g_4 \text{ in } 0 < x < L, \qquad w = w_x = 0 \text{ at } x = 0, L$$

which admits a unique solution $w \in H^3 \cap H_0^1(I)$ because $\left(I - \frac{\partial^2}{\partial x^2}\right) g_4 \in H^{-1}(I)$.

Thus, $\widetilde{A}$ is indeed the infinitesimal generator of a semigroup of operators in H.

In order to prove local existence of problem (1.4)-(1.6) it is enough to prove that $D^{-1}N(U)$ is locally Lipschitz continuous in H.

Let $U = (u, y, w, p)^\tau$ and $\tilde{U} = (\tilde{u}, \tilde{y}, \tilde{w}, \tilde{p})^\tau$ be elements of H. A direct calculation shows that

$$D^{-1}[N(U) - N(\tilde{U})] = (0, \tilde{f}, 0, \tilde{g})^\tau$$

where

$$\tilde{f} = \frac{2}{(1-\mu)\epsilon}\left[\frac{1}{2}(w_x^2 - \tilde{w}_x^2) + k_1(w - \tilde{w})\right]_x + \epsilon^{\alpha-1}(\tilde{y} - y)$$

and

$$\tilde{g} = \left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\left\{\frac{2}{1-\mu}\left[(u_x + \frac{1}{2}w_x^2 + k_1 w)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right]_x\right.$$
$$\left. - (p - \tilde{p}) + (p_{xx} - \tilde{p}_{xx}) + \frac{2k_1}{1-\mu}\left[\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w} - u_x - \frac{1}{2}w_x^2 - k_1 w\right]\right\}.$$

We have to estimate the expression

$$\|D^{-1}[N(U) - N(\tilde{U})]\|_H^2 = \epsilon\|\tilde{f}\|^2 + \|\tilde{g}\|^2 + \|\tilde{g}_x\|^2.$$

Assuming that $k_1 \in H^1(I)$ we can easily prove that

$$\begin{aligned}\|\tilde{f}\|^2 \;\le\; & \frac{C}{\epsilon^2}\{\|w_x - \tilde{w}_x\|_\infty^2(\|w_{xx}\| + \|\tilde{w}_{xx}\|)^2 \\ + \; & \|w_{xx} - \tilde{w}_{xx}\|^2(\|w_x\|_\infty + \|\tilde{w}_x\|_\infty)^2\} + C\epsilon^{2(\alpha-1)}\|y - \tilde{y}\|^2 \\ + \; & \frac{C}{(1-\mu)^2\epsilon^2}\|k_1\|_{H^1}^2\|w_x - \tilde{w}_x\|^2.\end{aligned}$$

Using the embedding $H^1(I) \hookrightarrow L^\infty(I)$ we deduce from the above estimate that

$$\|\tilde{f}\| \leq C(1 + \|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H, \tag{2.4}$$

where C is a positive constant depending on ϵ, μ, α and $\|k_1\|_{H^1}$.

Now, let us estimate $\|\tilde{g}\|_{H^1(I)}$. First, let

$$g_1 = \frac{2}{1-\mu}\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\left[\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right]_x$$

Taking into account that the operator $\left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}$ is bounded from $L^2(I)$ into $H_0^1(I)$ we deduce that

$$\|g_1\|_{H^1} \leq C\left\|\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)w_x - \left(\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w}\right)\tilde{w}_x\right\|. \tag{2.5}$$

Adding and substracting the term $\left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)\tilde{w}_x$ inside the norm on the right hand side of (2.5) it is easy to see that

$$\|g_1\|_{H^1} \leq C(\mu, \epsilon, \|k_1\|_{H^1})(\|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H.$$

Finally, let g_2 be given by

$$\begin{aligned} g_2 = \left(I - \frac{\partial^2}{\partial x^2}\right)^{-1}\Bigg\{ &- (p - \tilde{p}) + p_{xx} - \tilde{p}_{xx} + \\ &+ \frac{2k_1}{1-\mu}\left[\tilde{u}_x + \frac{1}{2}\tilde{w}_x^2 + k_1\tilde{w} - u_x - \frac{1}{2}w_x^2 - k_1 w\right]\Bigg\}. \end{aligned}$$

A similar discussion allows to show that

$$\|g_2\|_{H^1} \leq C(\mu, \epsilon, \|k_1\|_{H^1})(\|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H. \tag{2.6}$$

From (2.4), (2.5) and (2.6) we deduce that

$$\|D^{-1}[N(U) - N(\tilde{U})]\|_H \leq C(1 + \|U\|_H + \|\tilde{U}\|_H)\|U - \tilde{U}\|_H$$

where C is a positive constant depending on ε, μ, α and $\|k_1\|_{H^1}$. This proves that $D^{-1}N(U)$ is locally Lipschitz continuous in H.

Consequently, one obtains local existence of a unique finite energy solution.

Global existence in our case is consequence of the energy identity (1.8) which provides a priori bounds in the energy space for all $t \geq 0$.

We have shown:

Theorem 2.1. *Let* $\epsilon > 0$, $0 \leq \alpha$, $0 < \mu < 1$, $k_1 \in H^1(I)$ *and* $(u_0, u_1, w_0, w_1) \in H$. *Then, problem* (1.4)–(1.6) *has a unique global (weak) solution*

$$(u^\epsilon, u_t^\epsilon, w^\epsilon, w_t^\epsilon) \in C([0, +\infty); H)$$

and the total energy $E_\epsilon(t)$ *given by* (1.7) *satisfies* (1.8) *for all* $t \geq 0$.

3 The asymptotic limit

In this section we study the asymptotic limit of the solution $\{u^\epsilon, w^\epsilon\}$ of (1.4)-(1.6) es $\epsilon \to 0^+$.

Let $\epsilon > 0$, $0 < \alpha$ and $0 < \mu < 1$.

From the energy dissipation law (1.8) that guarantees that $E_\epsilon(t) \leq E_\epsilon(0)$ for all $t \geq 0$ and all ϵ, we deduce that the sequences

$$\{\sqrt{\epsilon} u_t^\epsilon\}, \left\{u_x^\epsilon + \frac{1}{2}(w_x^\epsilon)^2 + k_1 w^\epsilon\right\}, \{w_t^\epsilon\}, \{w_{xt}^\epsilon\} \text{ and } \{w_{xx}^\epsilon\}$$

are bounded in $L^\infty(0, +\infty; L^2(I))$ and

$$\{\epsilon^{\alpha/2} u_t^\epsilon\}, \{w_t^\epsilon\} \text{ and } \{w_{xt}^\epsilon\}$$

are bounded in $L^2(0, +\infty; L^2(I))$.

Extracting subsequences (that we still denote by the index ε in order to simplify notations) we deduce that there exist $\xi(x, t)$, $\eta(x, t)$ and $z(x, t)$ such that

$$\sqrt{\epsilon} u_t^\epsilon \rightharpoonup \xi \quad \text{weakly * in } L^\infty(0, +\infty; L^2(I)) \tag{3.1}$$

$$u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon \rightharpoonup \eta \quad \text{weakly * in } L^\infty(0, +\infty; L^2(I)) \tag{3.2}$$

and

$$w^\varepsilon \rightharpoonup z \quad \text{weakly * in } L^\infty(0, +\infty; H^2(I)) \cap W^{1,\infty}(0, +\infty; H_0^1(I)) \tag{3.3}$$

as $\epsilon \to 0$.

Clearly, the weak convergence in (3.3) is enough to allow us to pass to the limit in the linear part of the equation for w^ϵ in (1.4) provided, say, $k_1 \in L^\infty(I)$.

It remains to identify the weak limit of the nonlinear terms $\{u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2\}$ and $\left[\left(u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon\right) w_x^\epsilon\right]_x$ as $\epsilon \to 0$.

As we said above, the boundedness of $E_\epsilon(t)$ implies that $\{w^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^\infty(0,\infty;H_0^2(I))\cap W^{1,\infty}(0,+\infty;H_0^1(I))$. Then, we can use Aubin-Lions compactness lemma [4] to deduce that

$$w^\epsilon \to z \quad \text{strongly in } L^\infty(0,T;H^{2-\delta}(I)) \tag{3.4}$$

as $\varepsilon \to 0$ for any $\delta > 0$ and $T < +\infty$.

Combining (3.2) with (3.4) we deduce that

$$\left(u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon\right) w_x^\epsilon \rightharpoonup \eta z_x \quad \text{weakly in } L^2(I\times(0,T))$$

as $\epsilon \to 0$ for any $T < +\infty$.

Let us find out what the value of η is. We claim that

a) η is independent of x and

b) η is given by

$$\eta = \frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx.$$

To see this we first observe that $\left\{u_x^\epsilon\right\}$ is bounded in $L^2(I\times(0,T))$ since

$$\begin{aligned}
\int_0^L \left(u_x^\epsilon\right)^2 dx &= \int_0^L \left[u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon - \frac{1}{2}\left(w_x^\epsilon\right)^2 - k_1 w^\epsilon\right]^2 dx \\
&\le C\left[E_\epsilon(0) + \int_0^L \left(w_x^\epsilon\right)^4 dx + \int_0^L k_1^2\left(w^\epsilon\right)^2 dx\right] \\
&\le C\left[E_\epsilon(0) + \left(\int_0^L \left(w_{xx}^\epsilon\right)^2 dx\right)^2 + \|k_1\|_\infty^2 \int_0^L \left(w^\epsilon\right)^2 dx\right] \\
&\le C E_\epsilon(0)
\end{aligned}$$

for some positive constant C depending on the initial energy $E_\epsilon(0)$ and k_1. Obviously, this constant is independent of ϵ.

Thus, there exists a subsequence such that

$$u_x^\epsilon \rightharpoonup \rho \quad \text{weakly in } L^2(I\times(0,T)) \tag{3.5}$$

as $\epsilon \to 0$ for some $\rho = \rho(x,t)$. Using (3.4) and (3.5) we deduce that

$$u_x^\epsilon + \frac{1}{2}\left(w_x^\epsilon\right)^2 + k_1 w^\epsilon \rightharpoonup \rho + \frac{1}{2}z_x^2 + k_1 z = \eta \tag{3.6}$$

as $\epsilon \to 0$, weakly in $L^2(I \times (0, T))$.

Since $\alpha > 0$, using Poincaré's inequality and (1.8) we can bound $\{u^\epsilon\}$ in $L^\infty(0, T; H_0^1(I))$ to obtain that

$$\epsilon^\alpha u_t^\epsilon \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0, T; H_0^1(I)) \tag{3.7}$$

as $\epsilon \to 0$. Now, using (3.1) we also know that

$$\epsilon u_{tt}^\epsilon = \sqrt{\epsilon}\sqrt{\epsilon} u_{tt}^\epsilon \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0, T; L^2(I)) \tag{3.8}$$

as $\epsilon \to 0$. Thus, from the first equation in (1.4), (3.2) (3.7) and (3.8) we obtain that

$$\eta_x = \left[\rho + \frac{1}{2} z_x^2 + k_1 z\right]_x = 0$$

therefore, $\eta = \eta(t)$ which proves claim a).

To prove item b) we integrate the identity $\eta = \rho + \frac{1}{2} z_x^2 + k_1 z$ in x from $x = 0$ up to $x = L$ to obtain

$$L\eta(t) = \int_0^L \rho dx + \frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx = \frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx$$

since $\int_0^L \rho dx = \lim_{\epsilon \to 0} \int_0^L u_x^\epsilon dx = 0$, because u^ϵ vanishes at the boundary $x = 0, L$. Consequently,

$$\eta(t) = \frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx.$$

The above discussion indicates that

$$\left[\left(u_x^\epsilon + \frac{1}{2}(w_x^\epsilon)^2 + k_1 w^\epsilon\right) w_x^\epsilon\right]_x \rightharpoonup \left(\frac{1}{2L}\int_0^L z_x^2 dx + \frac{1}{L}\int_0^L k_1 z dx\right) z_{xx}$$

as $\epsilon \to 0$, weakly in $L^2(0, T; H^{-1}(I))$.

We conclude that the component w^ϵ in system (1.4)-(1.6) converges to the solution $z = z(x, t)$ of (1.9) weakly in $L^2(0, T; H_0^2(I))$ as $\epsilon \to 0$ for any $T < +\infty$.

Clearly z satisfies the boundary conditions in (1.11).

Finally we want to identify the initial data of the limit system. Since $w^\epsilon \to z$ in $C([0, T]; H^{2-\delta}(I))$ as $\epsilon \to 0$ for any $T < +\infty$ then $w^\epsilon(x, 0) \to z(x, 0)$ as $\epsilon \to 0$ in $H^{2-\delta}(I)$. Hence $z(x, 0) = w_0(x)$. Observing that

$$\{w_t^\varepsilon\} \text{ is bounded in } L^\infty(0, T; H_0^1(I))$$
$$\{w_{tt}^\varepsilon\} \text{ is bounded in } L^\infty(0, T; L^2(I))$$

for any $T < +\infty$ (the last bound is easily obtained using the equation in (1.4) that w^ϵ satisfies and our previous discussion), from (3.9) and using Aubin-Lions compactness lemma [4] it follows that $w_t^\epsilon \to z_t$ in $C([0,T]; L^2(I))$ as $\epsilon \to 0$. In particular, $w_t^\epsilon(x,0) \to z_t(x,0)$ as $\epsilon \to 0$ in $L^2(I)$. Hence $z_t(x,0) = w_1(x)$.

The above results can be summarized as follows.

Theorem 3.1. *Let* $(u_0, u_1, w_0, w_1) \in H = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I)$, $0 < \mu < 1, \alpha > 0$ *and* $k_1 \in H^1(I)$. *Consider the global solution* u^ϵ, w^ϵ *of system* (1.4)–(1.6) *obtained in Theorem* 2.1. *Then, as* $\epsilon \to 0^+$,

$$w^\epsilon \rightharpoonup z \quad \text{weakly in} \quad L^2(0,T; H_0^2(I))$$

Furthermore,

$$U_x^\epsilon \rightharpoonup \frac{1}{2L}\int_0^L (z_x^2 + 2k_1 z)dx - \frac{1}{2}z_x^2 - k_1 z$$

weakly in $L^2(I \times (0,T))$ *as* $\epsilon \to 0$ *for any* $T < +\infty$, *where* $z = z(x,t)$ *is the global (weak) solution of problem* (1.9)–(1.11).

4 Uniform stabilization as $\varepsilon \to 0$

The total energy of the limit system (1.9)–(1.11) is given by

$$G(t) = \frac{1}{2}\int_0^L \left(z_t^2 + z_{xx}^2 + z_{xt}^2\right) dx + \frac{1}{(1-\mu)L}\left(\frac{1}{2}\int_0^L z_x^2 dx + \int_0^L k_1 z dx\right)^2$$

and it is dissipated according to the law

$$\frac{dG(t)}{dt} = -\int_0^L (z_t^2 + z_{xt}^2)dx.$$

Then, it is not difficult to prove that $G(t)$ decays exponentially as $t \to +\infty$.

In this Section we prove that the energy $E_\epsilon(t)$ associated to problem (1.4)–(1.6) also decays exponentially as $t \to \infty$ and that the decay rate is uniform (as $\epsilon \to 0$) provided $0 \le \alpha \le 1$, recovering the rate of decay of the limit system.

More precisely, the following holds:

Theorem 4.1. *Let u^ϵ, w^ϵ be the global solution of system* (1.4)–(1.6) *obtained in Theorem* 2.1 *with* $0 \leq \alpha \leq 1$. *Then, there exist positive constants* $C > 0$ *and* $\beta > 0$ *such that*

$$E_\epsilon(t) \leq C E_\epsilon(0) \, \exp\left(-\frac{\beta t}{1 + \epsilon^\alpha [E_\epsilon(0) + \|k_1\|_\infty^2]}\right)$$

for all $t \geq 0$ *and all* $0 < \epsilon < 1$.

Proof. Let $\epsilon > 0$. In order to simplify notations we write $u^\epsilon = u$, $w^\epsilon = w$. We consider the functional

$$F_\epsilon(t) = \epsilon \int_0^L u u_t dx + \int_0^L (w w_t + w_x w_{xt}) dx. \tag{4.1}$$

Direct calculations using the equations give us that

$$\begin{aligned} \frac{dF_\varepsilon}{dt} \leq &- \frac{8}{1-\mu} \int_0^L \left(u_x + \frac{1}{2} w_x^2 + k_1 w\right)^2 dx - \varepsilon^\alpha \int_0^L u u_t dx \\ &+ \varepsilon \int_0^L u_t^2 dx - \int_0^L w_{xx}^2 dx - \int_0^L w w_t dx + \int_0^L w w_{xxt} dx \\ &+ \int_0^L [w_t^2 + w_{xt}^2] dx. \end{aligned} \tag{4.2}$$

In the following estimates C denotes a positive constant which may vary from line to line but is independent of ϵ.

For any $\gamma > 0$ we have

$$\left| \int_0^L w w_{xxt} dx \right| = \left| \int_0^L w_x w_{xt} dx \right| \leq C \int_0^L \left[\gamma w_{xx}^2 + \frac{1}{\gamma} w_{xt}^2 \right] dx \tag{4.3}$$

$$\left| \int_0^L w w_t dx \right| \leq C \int_0^L \left[\gamma w_{xx}^2 + \frac{1}{\gamma} w_t^2 \right] dx, \tag{4.4}$$

since $\| w_{xx} \|$ defines a norm in $H^2 \cap H_0^1(I)$ which is equivalent to the one induced by $H^2(I)$.

Also

$$\epsilon^\alpha \left| \int_0^L u u_t dx \right| \leq \frac{\epsilon^\alpha}{2\gamma} \int_0^L u_t^2 dx + \frac{\epsilon^\alpha \gamma}{2} \int_0^L u^2 dx. \tag{4.5}$$

Moreover

$$\int_0^L u^2 dx \le C \int_0^L u_x^2 dx$$
$$\le C\left\{ \int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx + \left(\int_0^L w_{xx}^2 dx\right)^2 + \|k_1\|_\infty^2 \int_0^L w_{xx}^2 dx \right\} \tag{4.6}$$
$$\le C\left\{ \int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx + C\big[E_\epsilon(0) + \|k_1\|_\infty^2\big] \int_0^L w_{xx}^2 dx \right\}.$$

Consequently, from (4.5) and (4.6) we obtain that

$$\epsilon^\alpha \left| \int_0^L u u_t dx \right| \le \frac{\epsilon^\alpha}{2\gamma} \int_0^L u_t^2 dx + \frac{c\epsilon^\alpha \gamma}{2} \int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx$$
$$+ \frac{c\epsilon^\alpha \gamma}{2}\big[E_\epsilon(0) + \|k_1\|_\infty^2\big] \int_0^L w_{xx}^2 dx. \tag{4.7}$$

Let $\delta > 0$ and define $G_{\varepsilon,\delta}(t)$ given by

$$G_{\epsilon,\delta}(t) = E_\epsilon(t) + \delta F_\epsilon(t)$$

Using (1.8) and (4.2) together with (4.3)–(4.7) we obtain that

$$\begin{aligned} \frac{dG_{\varepsilon,\delta}(t)}{dt} \le & -\left\{\varepsilon^{\alpha-1} - \frac{\varepsilon^{\alpha-1}\delta}{2\gamma} - \delta\right\}\varepsilon \int_0^L u_t^2 dx \\ & - \left\{1 - \delta - \frac{\delta C}{\gamma}\right\} \int_0^L [w_t^2 + w_{xt}^2] dx \\ & - \delta\left\{\frac{8}{1-\mu} - \frac{C\varepsilon^\alpha \gamma}{2}\right\} \int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx \\ & - \delta\left\{1 - \gamma C\left[2 + \frac{\varepsilon^\alpha}{2}\{E_\varepsilon(0) + \|k_1\|_\infty^2\}\right]\right\} \int_0^L w_{xx}^2 dx. \end{aligned} \tag{4.8}$$

Now let us choose $\gamma > 0$ as

$$\gamma = \lambda\left[2 + \frac{\epsilon^\alpha}{2}\{E_\epsilon(0) + \|k_1\|_\infty^2\}\right]^{-1}$$

where $\lambda > 0$ is small enough but independent of ε and $E_\epsilon(0)$. Then, (4.8) reads as follows

$$
\begin{aligned}
\frac{dG_{\epsilon,\delta}}{dt} \leq & -\left\{\epsilon^{\alpha-1} - \frac{\epsilon^{\alpha-1}\delta}{2\lambda}\left(2+\frac{\epsilon^{\alpha}}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right) - \delta\right\}\epsilon\int_0^L u_t^2 dx \\
& - \left\{1-\delta-\frac{\delta C}{\lambda}\left(2+\frac{\epsilon^{\alpha}}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)\right\}\int_0^L [w_t^2 + w_{xt}^2]dx \\
& - \delta\left\{\frac{8}{1-\mu} - \frac{c\epsilon^{\alpha}\lambda}{2\left(2+\frac{\epsilon^{\alpha}}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)}\right\} \\
& \int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx - \delta\{1-\lambda C\}\int_0^L w_{xx}^2 dx.
\end{aligned}
\tag{4.9}
$$

We want to impose suitable conditions on δ (and λ) so that the coefficients on the right hand side of (4.9) are all strictly less that $-\frac{\delta}{2}$. We will do this in case when $k_1 \not\equiv 0$ since the situation $k_1 \equiv 0$ was already treated in [6].

We choose $\lambda > 0$ small so that

$$
\lambda < \min\left\{\frac{8\|k_1\|_\infty^2}{1-\mu}, \frac{1}{C}\right\}
$$

which implies that $1-\lambda C > 0$ and

$$
\frac{8}{1-\mu} - \frac{C\epsilon^{\alpha}\lambda}{2\left(2+\frac{\epsilon^{\alpha}}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)} > 0.
$$

Once this choice of λ is done we need $\delta > 0$ to satisfy

$$
\delta \leq \frac{\epsilon^{\alpha-1}}{\frac{3}{2} + \frac{\epsilon^{\alpha-1}}{2\lambda}\left(2+\frac{\epsilon^{\alpha}}{2}[E_\epsilon(0)+\|k_1\|_\infty^2]\right)} \tag{4.10}
$$

and

$$
\delta \leq \frac{1}{2}\left[1+\frac{C}{\lambda}\left(2+\frac{\epsilon^{\alpha}}{2}\{E_\epsilon(0)+\|k_1\|_\infty^2\}\right)\right]^{-1}. \tag{4.11}
$$

We observe that (4.10) and (4.11) will be satisfied if we choose $\delta > 0$ of the form

$$
\delta = C_1\{1+\epsilon^{\alpha}[E_\epsilon(0)+\|k_1\|_\infty^2]\}^{-1}
$$

for some positive constant C_1 (that may depend of λ) but is independent of $0 < \epsilon < 1$. With this choice, the coefficients of $\epsilon \int_0^L u_t^2 dx$ and $\int_0^L [w_t^2 + w_{xt}^2] dx$ on the right hand side of (4.9) are, respectively, less than or equal than $-\delta/2$ and $-1/2$.

In conclusion, with the above choice of λ and $\delta > 0$, (4.9) implies that

$$\frac{dG_{\epsilon,\delta}}{dt} \leq -\min\left\{\frac{1}{2}, \frac{\delta}{2}\right\} E_\varepsilon(t). \tag{4.12}$$

Finally, we compare $E_\epsilon(t)$ with $G_{\epsilon,.\delta}(t)$. Using (4.1) together with (4.3), (4.4) and (4.7) we obtain that

$$\begin{aligned}
|F_\epsilon(t)| &\leq \frac{\epsilon}{2}\int_0^L u_t^2 dx + \frac{C\epsilon}{2}\int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2 dx \\
&+ C\int_0^L (w_t^2 + w_{xt}^2 + w_{xx}^2) dx + \frac{C\epsilon}{2}[E_\epsilon(0) + \|k_1\|_\infty^2]\int_0^L w_{xx}^2 dx \\
&\leq (C\epsilon + C + C\epsilon[E_\epsilon(0) + \|K_1\|_\infty^2]) E_\epsilon(t) \\
&\leq \tilde{C}(1 + \epsilon[E_\epsilon(0) + \|k_1\|_\infty^2]) E_\epsilon(t)
\end{aligned}$$

where $\tilde{C}$ is a positive constant independent of $0 < \epsilon < 1$. Thus,

$$\begin{aligned}
|G_{\epsilon,\delta}(t) - E_\epsilon(t)| = \delta|F_\epsilon(t)| &\leq \delta\tilde{C}[1 + E_\epsilon(0) + \|k_1\|_\infty^2] E_\epsilon(t) \\
&\leq \delta\tilde{C} E_\epsilon(t)
\end{aligned} \tag{4.13}$$

for some positive constant $\tilde{C}$ depending only on the initial data and $\|k_1\|_\infty^2$ (since $E_\epsilon(0)$ is bounded in ϵ).

Then, (4.13) together with (4.12) and our choice of δ implies the conclusion of Theorem 2.

5 Final remarks and comments

When $\alpha = 0$ the global well-posedness of (1.4)–(1.6) is still valid for each $\epsilon > 0$ but, in this case, the asymptotic limit as $\epsilon \to 0$ is of a different nature. In fact, when $\alpha = 0$ the limit system is of the form

$$\begin{cases}
v_t = \dfrac{2}{1-\mu}\left[v_x + \dfrac{1}{2}z_x^2 + k_1 z\right]_x \\
z_{tt} + z_{xxxx} - z_{xxtt} = \dfrac{\partial}{\partial x} f(v, z) - g(v, z) - z_t + z_{xxt}
\end{cases} \tag{5.1}$$

for $0 < x < L, t > 0$. System (5.1) has initial conditions

$$v(x,0) = u_0(x), z(x,0) = w_0(x), z_t(x,0) = w_1(x), \quad 0 < x < L \tag{5.2}$$

and boundary conditions

$$v(0,t) = v(L,t) = z(0,t) = z(L,t) = z_x(0,t) = z_x(L,t) = 0. \tag{5.3}$$

System (5.1)–(5.3) is the coupling between a parabolic equation and a fourth order hyperbolic equation, thus it has a similar structure to a system of thermoelasticity. The total energy associated with (4.1) is given by

$$E(t) = \frac{1}{2}\int_0^L \left\{ z_t^2 + z_{xx}^2 + z_{xt}^2 + \left(v_x + \frac{1}{2}z_x^2 + k_1 z \right)^2 \right\} dx$$

and satisfies

$$\frac{dE}{dt} = -\int_0^L (v_t^2 + z_t^2 + z_{xt}^2)\, dx.$$

According to the discussion of Theorem 4.1 we can pass to the limit as $\varepsilon \to 0$ to obtain the following decay property for the solution of (4.1)–(4.3)

$$E(t) \le CE(0) \exp\left(-\frac{\beta t}{1 + E(0) + \|k_1\|_\infty^2} \right)$$

for all $t > 0$.

We refer to [6] for further developments of this issue in the case of the classical von Kármán and Timoshenko equations.

The analysis developed in this paper can be adapted to a variety of situations, including different boundary conditions. The interested reader is referred to [6] and [7] for the discussion of these issues in the case of von Kármán and Timoshenko equations.

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